PROBABILISTIC FINITE ELEMENTS (PFEM) APPLIED TO STRUCTURAL DYNAMICS AND FRACTURE MECHANICS

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OBJECTIVES OF PFEM PROGRAM:

- TO PROVIDE INTEGRATED METHODOLOGIES FOR PROBABILISTIC FINITE ELEMENTS BASED ON VARIATIONAL PRINCIPLES WHICH ARE COMPUTATIONALLY EFFICIENT
- TO INVESTIGATE FUNDAMENTAL ASPECTS OF IMPLEMENTATION OF PROBABILISTIC FINITE ELEMENTS IN TRANSIENT ANALYSIS
- TO DEVELOP BENCHMARK PROBLEMS AND SOLUTIONS

SLIDE 1

PROBABILISTIC FINITE ELEMENTS (PFEM) APPLIED TO STRUCTURAL DYNAMICS AND FRACTURE MECHANICS

Wing-Kam Liu, Ted Belytschko, A. Mani, and G. Besterfield

SLIDE 1

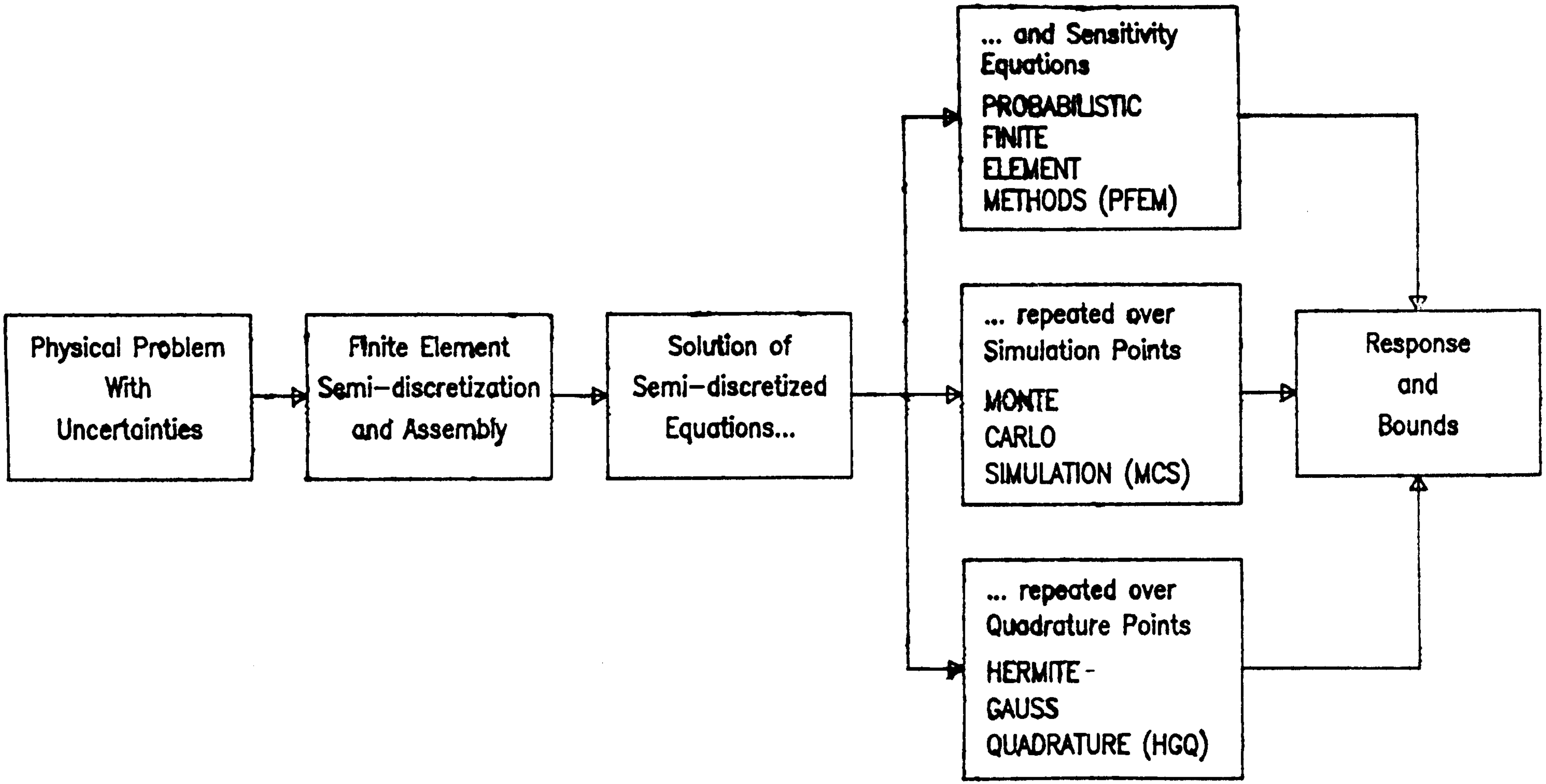
The purpose of this work is to develop computationally efficient methodologies for assessing the effects of randomness in loads, material properties, and other aspects of a problem by a finite element analysis. The resulting group of methods is called probabilistic finite elements (PFEM). The overall objective of this work is to develop methodologies whereby

1. the lifetime of a component can be predicted, accounting for the variability in the material and geometry of the component, the loads, and other aspects of the environment;
2. the range of response expected in a particular scenario can be presented to the analyst in addition to the response itself.

Emphasis in this work has been placed on methods which are not statistical in character, that is, they do not involve Monte Carlo simulations. The reason for this choice of direction is that Monte Carlo simulations of complex nonlinear response require a tremendous amount of computation.

The focus of our efforts so far has been on nonlinear structural dynamics. However, in the continuation of this project, emphasis will be shifted to probabilistic fracture mechanics so that the effect of randomness in crack geometry and material properties can be studied interactively with the effect of random load and environment. The ultimate goal of this effort will be to predict the behavior of cracks in such environments, which is an essential step towards lifetime prediction of structural components.

In addition, we are investigating how these methods should be implemented and developing benchmark problems.


Physical Problem With Uncertainties
Finite Element Semi-discretization and Assembly
Solution of Semi-discretized Equations...
... and Sensitivity Equations
PROBABILISTIC FINITE ELEMENT METHODS (PFEM)
... repeated over Simulation Points
MONTE CARLO SIMULATION (MCS)
... repeated over Quadrature Points
HERMITE-GAUSS QUADRATURE (HGQ)
Response and Bounds

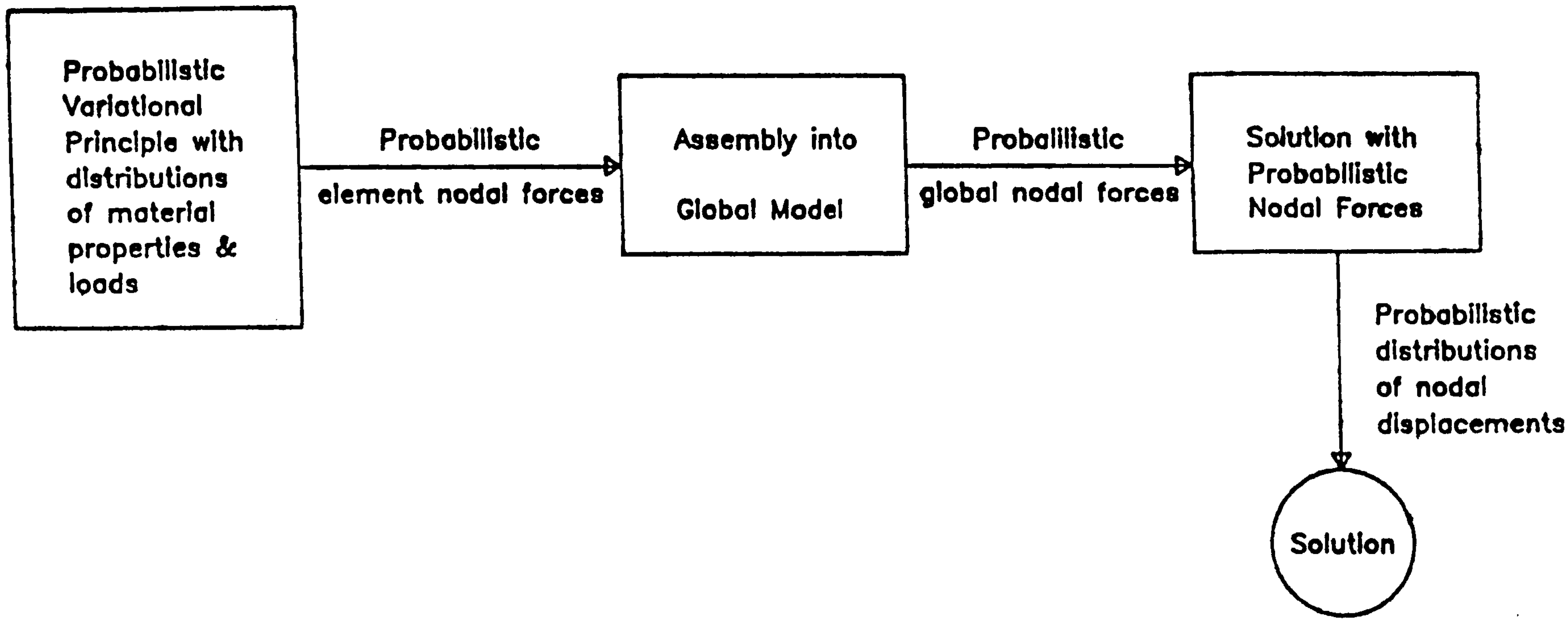
SLIDE 2

SLIDE 2

An outline of the schema in applying probabilistic finite elements is shown on this slide. The starting point is the physical problem which is characterized by uncertainties and the development of a finite element model. The latter involves the development of semidiscretized equations for the elements and their assembly into global equations. In the methods we have developed, the randomness of the problem is also described on an element basis, and the development of the global description of randomness is obtained by the same assembly procedure as for the deterministic variables.

The result of this assembly procedure is a set of semidiscretized equations, which, for a nonlinear structural dynamics system, are nonlinear algebraic equations. This slide shows three methods of approach to the solution of these equations.

1. Hermite-Gauss quadrature, which involves numerical quadrature over the probabilistic space which describes the problem; this method is feasible only for three to ten probabilistic functions.
2. Monte Carlo simulation, in which the semidiscretized equations are solved repeatedly with the values of the probabilistic variables in the model and environment determined by a random-number generator; this approach involves substantial computational effort.
3. Combination of the semidiscretized equation with sensitivity equations and some type of a perturbation method; this is the type of method which has been adopted in probabilistic finite elements, where a second-order moment method has been used.


Probabilistic
Variational
Principle with
distributions
of material
properties &
loads
Probabilistic
element nodal forces
Assembly into
Global Model
Probalilistic
global nodal forces
Solution with
Probabilistic
Nodal Forces
Probabilistic
distributions
of nodal
displacements
Solution

SLIDE 3

SLIDE 3

An essential element in obtaining semidiscretized equations is a variational principle which provides a weak form for the governing equations. At the right, we show the relationship between the variational principle and the solution. The variational principle is used to obtain the weak form, which, in terms of the computer architecture, leads to the probabilistic nodal forces. All of the deterministic and random properties are assembled in the same way from the element level to the global model. The solution of the semidiscretized system yields the probabilistic distributions of nodal displacements, which can then be returned to the element level to yield probabilistic distributions in strain, stress, and other response variables.

HU-WASHIZU VARIATIONAL FORM FOR PROBABILISTIC FINITE ELEMENT

$$\delta u_{iI} f_{iI} = \int_{\Omega_e} \Big[\delta\varepsilon_{ij} \underbrace{(C_{ijk\ell}\varepsilon_{k\ell} - \sigma_{ij})}_{\text{CONSTITUTIVE}} + \delta u_{i,j} \underbrace{\sigma_{ij} - \delta u_i b_i}_{\text{EQUILIBRIUM}} + \delta\sigma_{ij} \underbrace{(u_{i,j} - \varepsilon_{ij})}_{\text{STRAIN-DISPLACEMENT}} \Big] d\Omega$$

b_i . . . RANDOM DESCRIPTION OF MATERIAL, LOAD, BOUNDARY CONDITIONS $i = 1$ TO 9

$$\underset{\sim}{\underset{\sim}{M}}\underset{\sim}{a} + \underset{\sim}{f}(\underset{\sim}{b}, \underset{\sim}{d}, \dot{\underset{\sim}{d}}) = \underset{\sim}{F}(\underset{\sim}{b})$$

$\ddot{d}$

CD-87-30816

SLIDE 4

The starting point for the development of the probabilistic finite element method is the Hu-Washizu variational form. This variational form constitutes a weak form for the following equations:

1. the constitutive equations;
2. the equilibrium equations;
3. the strain displacement equations.

The randomness of the material and geometry in the component and in the environment is described by random variables b_i. We do not account for the effect of randomness on changes in inertial properties as reflected in the mass matrix $\underset{\sim}{M}$, so the structure of the semidiscretized probabilistic equations is as shown on the bottom of this slide. These equations are nonlinear ordinary differential equations; the nonlinearity arises from the nonlinear character of the internal forces, $\underset{\sim}{f}$.

SECOND MOMENT METHOD OPERATORS FOR FORMULATING PROBABILISTIC FINITE ELEMENTS

EXPECTED VALUE OPERATOR

$$E\left[\underset{\sim}{\Phi}(\underset{\sim}{b})\right] = \underbrace{\bar{\underset{\sim}{\Phi}}}_{\equiv \text{ MEAN}} + \underbrace{\frac{1}{2} \frac{\partial^2 \bar{\Phi}}{\partial b_i \, \partial b_j} \text{COV}(b_i, b_j)}_{\Delta \bar{\Phi}} \quad \text{SUM ON } i, j$$

OBTAINED BY TAYLOR SERIES

$$E\left[\underset{\sim}{f}\right] = \bar{f}$$

$$+ \frac{1}{2} \frac{\partial^2 \bar{f}}{\partial b_i \, \partial b_j} \text{COV}(b_i, b_j)$$

$$+ \left(\frac{\partial \bar{\underset{\sim}{C}}}{\partial b_i} \frac{\partial \bar{\underset{\sim}{v}}}{\partial b_j} + \frac{\partial \bar{K}}{\partial b_i} \frac{\partial \bar{d}}{\partial b_j} \right) \text{COV}(b_i, b_j)$$

$$+ \frac{1}{2} \left(\bar{\underset{\sim}{C}} \frac{\partial^2 \bar{\underset{\sim}{v}}}{\partial b_i \, \partial b_j} + \bar{\underset{\sim}{K}} \frac{\partial^2 \underset{\sim}{d}}{\partial b_i \, \partial b_j} \right) \text{COV}(b_i, b_j)$$

SLIDE 5

The development of the probabilistic finite element method is based on a second-order moment method. A fundamental aspect of this method is the representation of the probabilistic distribution by the mean value, which is denoted by superposed bars, and the covariance of the random variables. The expectation of any function of the probabilistic variables is given in the first expression on this slide.

The expectation of the internal nodal forces is obtained by a Taylor series and a linearization of the nonlinear equations of motion as indicated in the second equation. Here, $\bar{\utilde{K}}$ is a tangent stiffness matrix about the mean nonlinear path, $\bar{\utilde{C}}$ is a damping matrix, and $\bar{v}$ indicates the mean velocity history.

PROBABILISTIC FINITE ELEMENT GOVERNING EQUATIONS

MEAN VALUE EQUATION

$$\underset{\sim}{\underset{\sim}{M}}\bar{a} + \underset{\sim}{\bar{f}} = \underset{\sim}{F}$$

OR

$$\underset{\sim}{\underset{\sim}{M}}\bar{a} + \underset{\sim}{\underset{\sim}{\bar{C}}}\bar{v} + \underset{\sim}{\underset{\sim}{\bar{K}}}\bar{d} = \underset{\sim}{F}$$

SENSITIVITY EQUATIONS

$$\underset{\sim}{M} \frac{\partial \bar{a}}{\partial b_j} + \underset{\sim}{\bar{C}} \frac{\partial \bar{v}}{\partial b_j} + \underset{\sim}{\bar{K}} \frac{\partial \bar{d}}{\partial b_j} = \partial \underset{\sim}{f} / \partial b_j$$

VARIANCE EQUATION

$$\underset{\sim}{M}\, \Delta \underset{\sim}{\bar{d}} + \underset{\sim}{\bar{C}}\, \Delta \underset{\sim}{\bar{v}} + \underset{\sim}{K}\, \Delta \underset{\sim}{\bar{d}} = \Delta \bar{f}$$

$$\Delta \bar{\Phi} = \frac{1}{2} \sum_{j=1}^{9} \frac{\partial^2 \bar{\Phi}}{\partial b_i^2} \mathrm{VAR}(b_i)$$

SLIDE 6

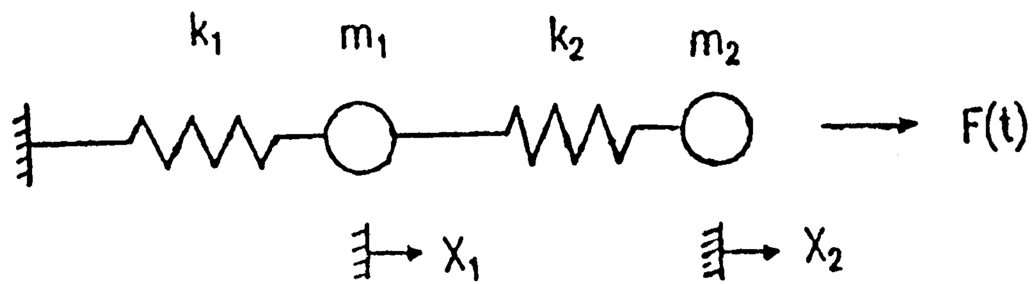
The governing equations for the system can then be written as follows. The mean value equation is a nonlinear equation which reflects the nonlinearities of the system. In addition, we have a system of sensitivity equations; the number of these equations is equal to the number of discrete random variables which define the probabilistic aspects of the component and its environment. Finally, we have the variance equation which gives the variance of the response variables in terms of the variance of the probabilistic variables.

ITEM	CALCULATED QUANTITIES	EQUATION NUMBERS	NUMBERS OF INTEGRATIONS in time
1	The mean values i.e. $\bar{\underline{d}}$, $\bar{\underline{v}}$ and $\bar{\underline{a}}$	(3.1a)	1
2	The sensitivity derivatives for each random variable b_j, $j = 1, \ldots, q$ i.e., $\dfrac{\partial \bar{\underline{d}}}{\partial b_j}$, $\dfrac{\partial \bar{\underline{v}}}{\partial b_j}$ and $\dfrac{\partial \bar{\underline{a}}}{\partial b_j}$	(3.2a)	q
3	The second order variations i.e., $\Delta\bar{\underline{d}}$, $\Delta\bar{\underline{v}}$ and $\Delta\bar{\underline{a}}$	(3.1b)	1
			Total = q + 2

SLIDE 7

SLIDE 7

The number of equations to be solved is summarized in this slide. The mean values are computed by a nonlinear equation in time, whereas the first-order variations are obtained by the q linearized sensitivity equations about the mean path. Finally, the second-order variations are obtained by a single equation in time. The total number of equations is then given by $q+2$, where q is the number of discrete random variables.



$F(t) = 25.0 \times 10^6 \sin(2000t)$

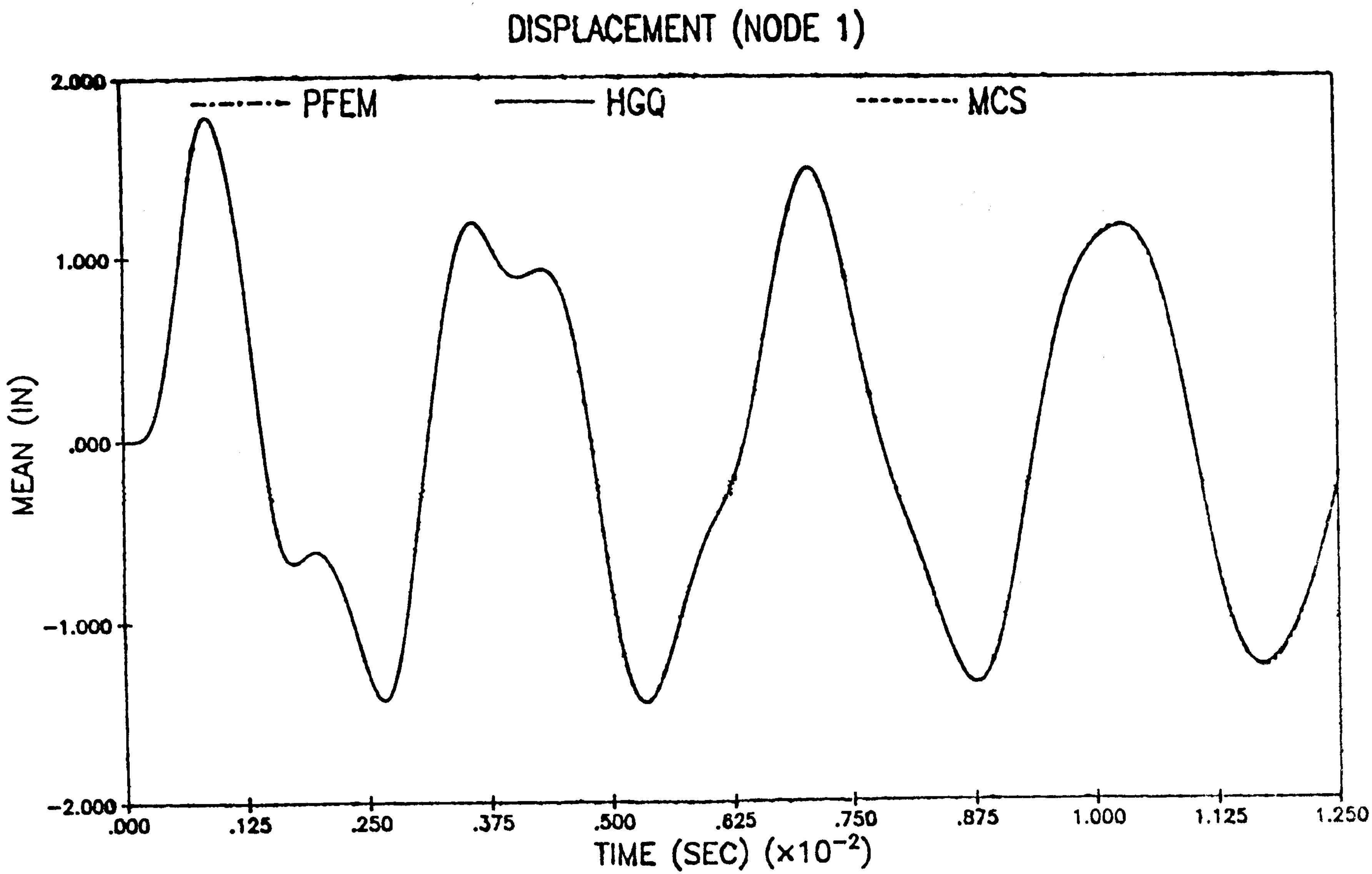
$$\left.\begin{array}{ll} m_1 = 0.372 & k_1 = 24.0 \times 10^6 \\ m_2 = 0.248 & k_2 = 12.0 \times 10^6 \end{array}\right]$$

Coefficient of Variation = 0.05

SLIDE 8

SLIDE 8

We will now consider several sample problems which have been solved in order to illustrate the type of output which is provided by the method and the capabilities of the method. The first example is a simple two-degree-of-freedom problem with the loading, mass, and stiffness given in the slide. The loading and the masses were considered to be deterministic, while the stiffness was assumed to be random with a coefficient of variation of 5%.


DISPLACEMENT (NODE 1)
PFEM
HGQ
MCS
2.000
1.000
.000
-1.000
-2.000
MEAN (IN)
.000
.125
.250
.375
.500
.625
.750
.875
1.000
1.125
1.250
TIME (SEC) (x10^-2)

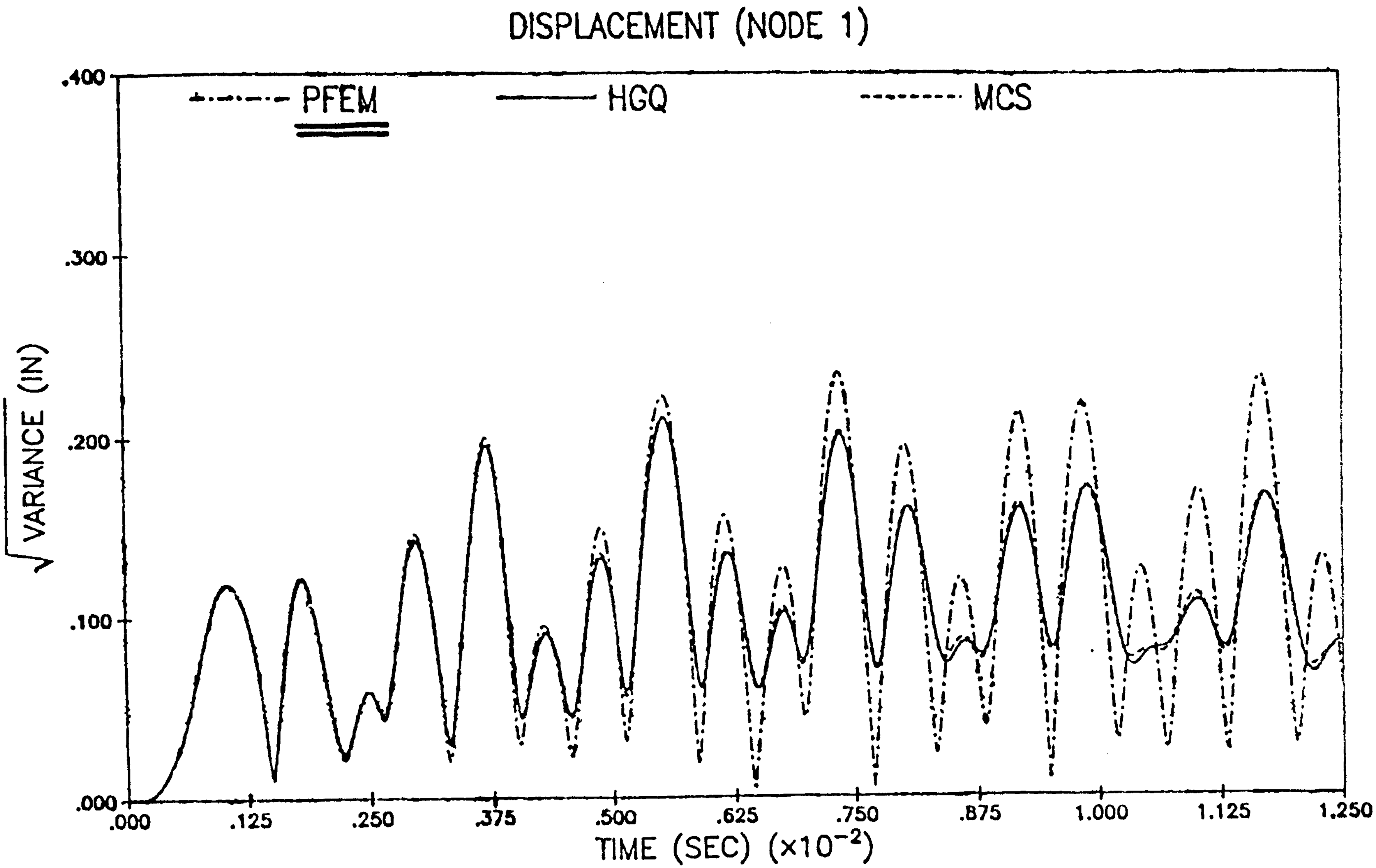
SLIDE 9

SLIDE 9

The two-degree-of-freedom problem was solved by three methods:

1. the probabilistic finite element method (PFEM);
2. Hermite-Gauss quadrature (HGQ);
3. Monte Carlo simulation (MCS).

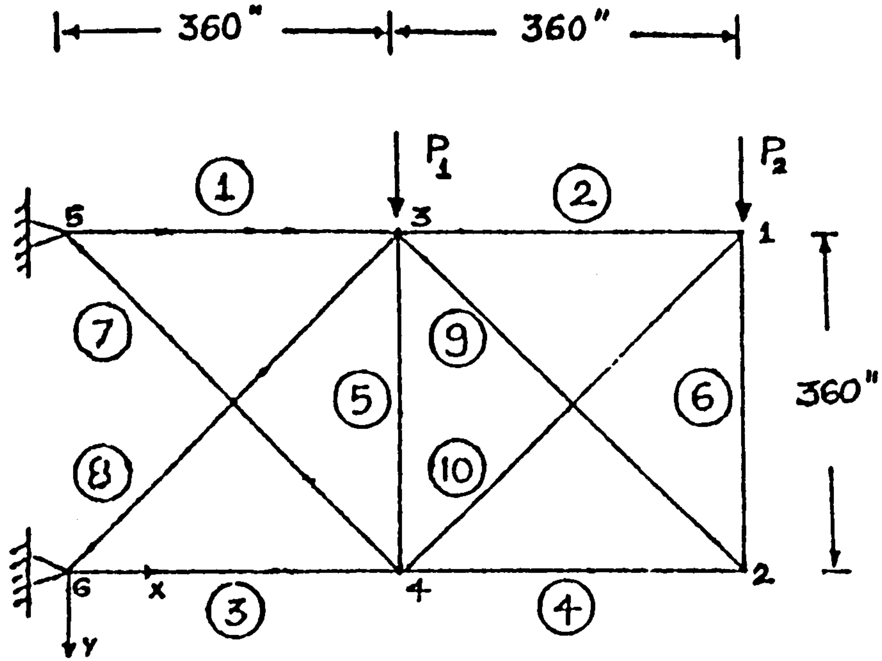
The reason for choosing the three methods was to evaluate the accuracy of the PFEM method. The PFEM method is much faster than the other two methods for large-scale problems, but it involves certain assumptions about the effects of probabilistic distributions in parameters which govern the component and its environment and their effect on the response. As can be seen from results given here, a mean displacement for node 1 as predicted by the probabilistic finite element method agrees exactly with the other two methods.

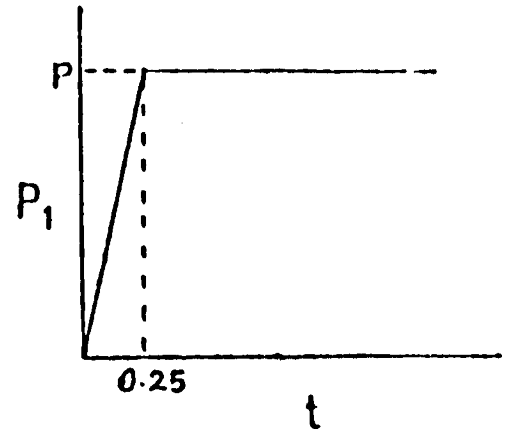

DISPLACEMENT (NODE 1)
PFEM
HGQ
MCS
√VARIANCE (IN)
.400
.300
.200
.100
.000
.000
.125
.250
.375
.500
.625
.750
.875
1.000
1.125
1.250
TIME (SEC) (×10⁻²)

SLIDE 10

SLIDE 10

This slide shows the variance in the displacement at node 1 as predicted by the three methods. In the variance, we see somewhat larger deviation in the three methods. The major reason for this is the effect of the canonical terms on the higher order equations. We have recently developed methods to ameliorate this effect, and these methods have been published in the forthcoming paper, "Probabilistic Transient Systems" (W. K. Liu, G. Besterfield, and T. Belytschko, Computer Methods in Applied Mechanics and Engineering, to appear).





$E = 30.0 \times 10^6$

$E_T = 30.0 \times 10^4$

$A = 6.0$

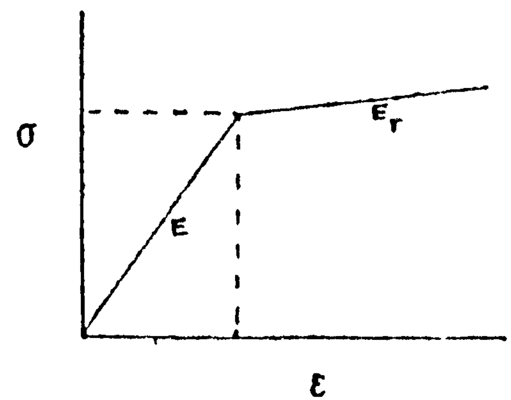
$\rho = 0.30$

$\sigma_Y = 15000.0$

$P = 175.0 \times 10^3$

$P_2 = 0.0$

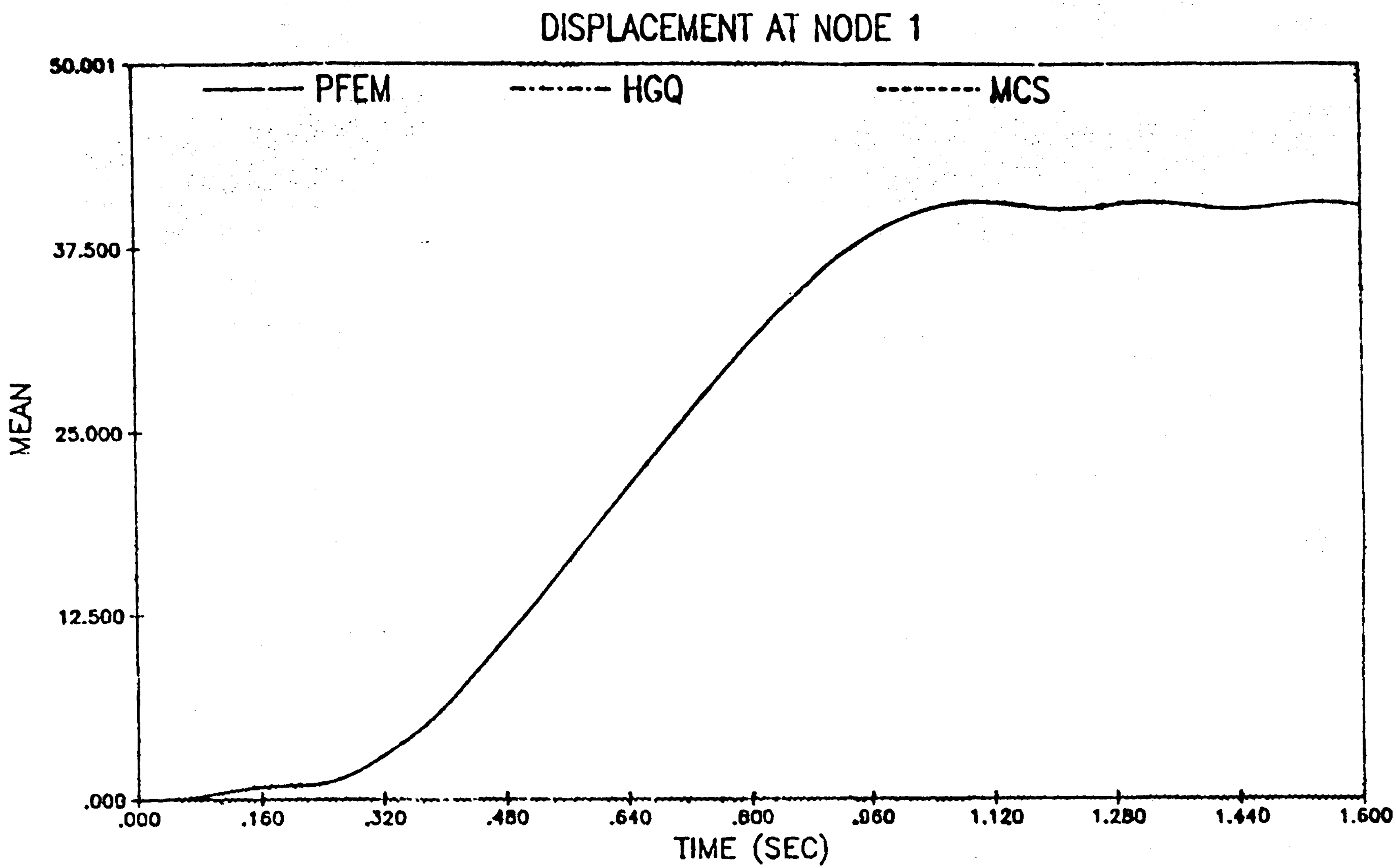
5% coef of variation in yield stress



SLIDE 11

SLIDE 11

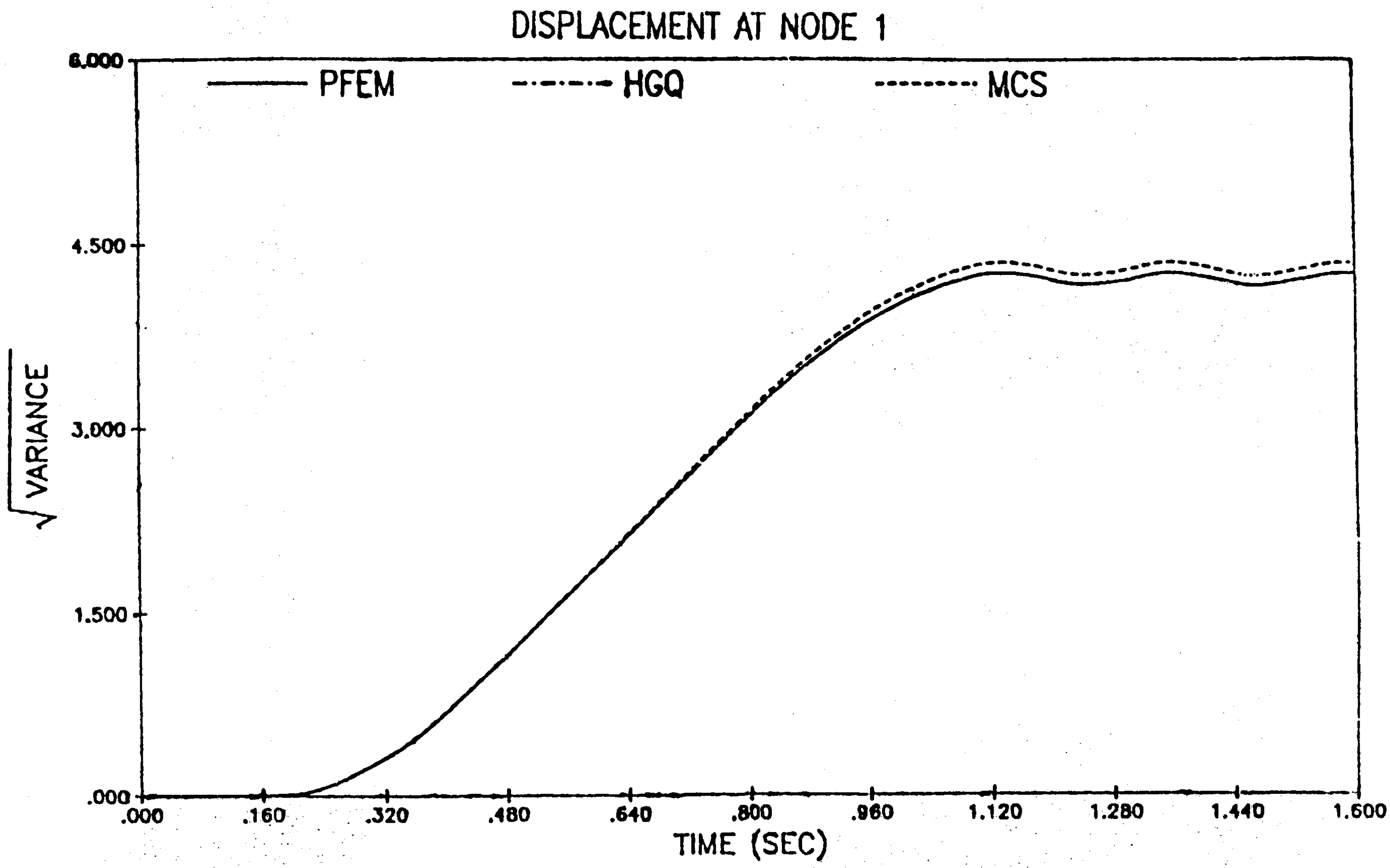
This example is far more complex and involves strong nonlinearities because the yield stress of the bars is assumed to be random and to have a coefficient of variation of 5%. The parameters of the problem are summarized on the figure.


DISPLACEMENT AT NODE 1
PFEM
HGQ
MCS
MEAN
50.001
37.500
25.000
12.500
.000
.000
.160
.320
.480
.640
.800
.960
1.120
1.280
1.440
1.600
TIME (SEC)

SLIDE 12

SLIDE 12

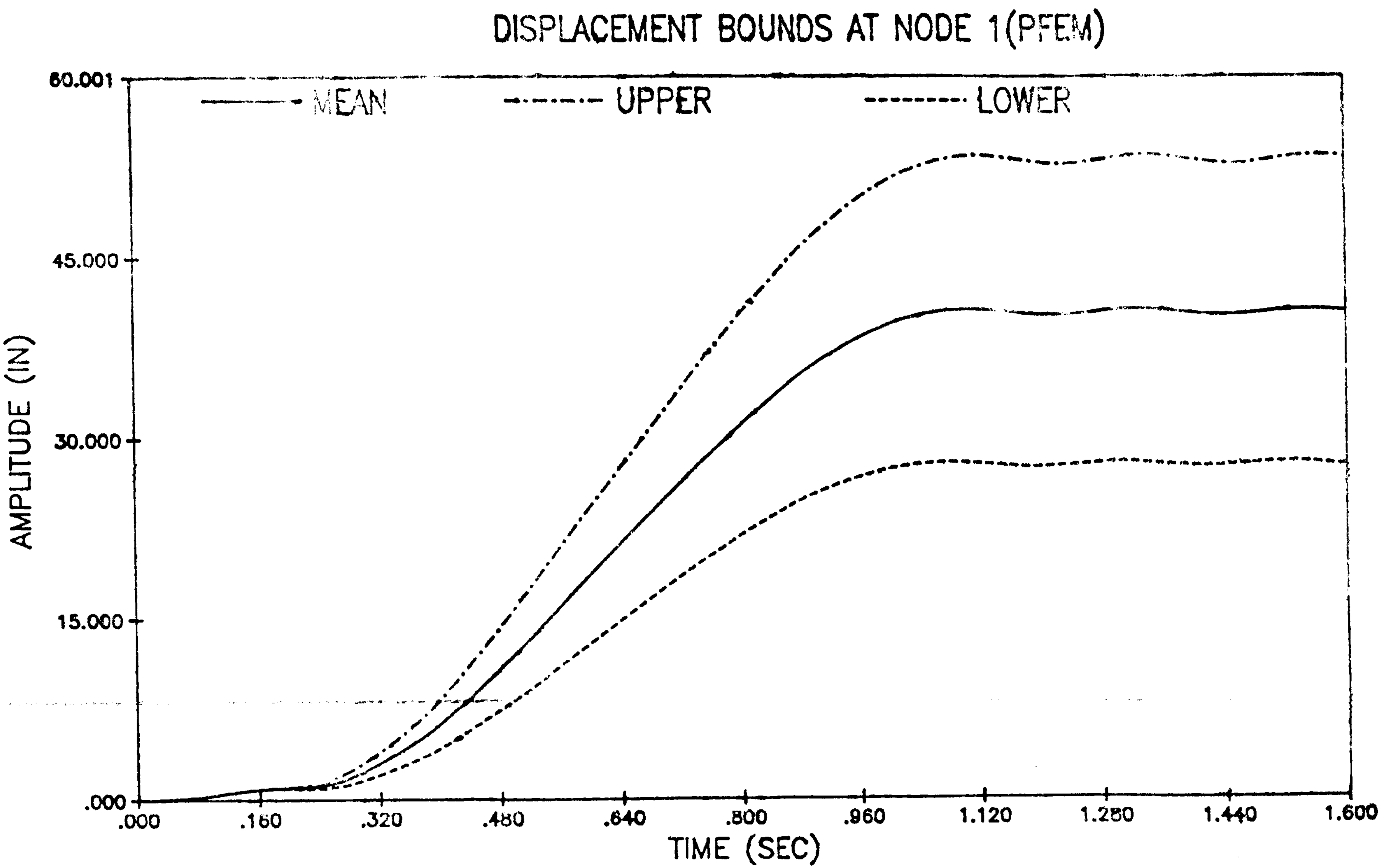
This shows the mean displacement at node 1, which again has been calculated by the probabilistic finite element method, Hermite-Gauss quadrature, and Monte Carlo simulation. As can be seen from the slide, even in this highly nonlinear problem, the three methods agree quite well.


DISPLACEMENT AT NODE 1
PFEM
HGQ
MCS
6.000
4.500
3.000
1.500
.000
√VARIANCE
.000
.160
.320
.480
.640
.800
.960
1.120
1.280
1.440
1.600
TIME (SEC)

SLIDE 13

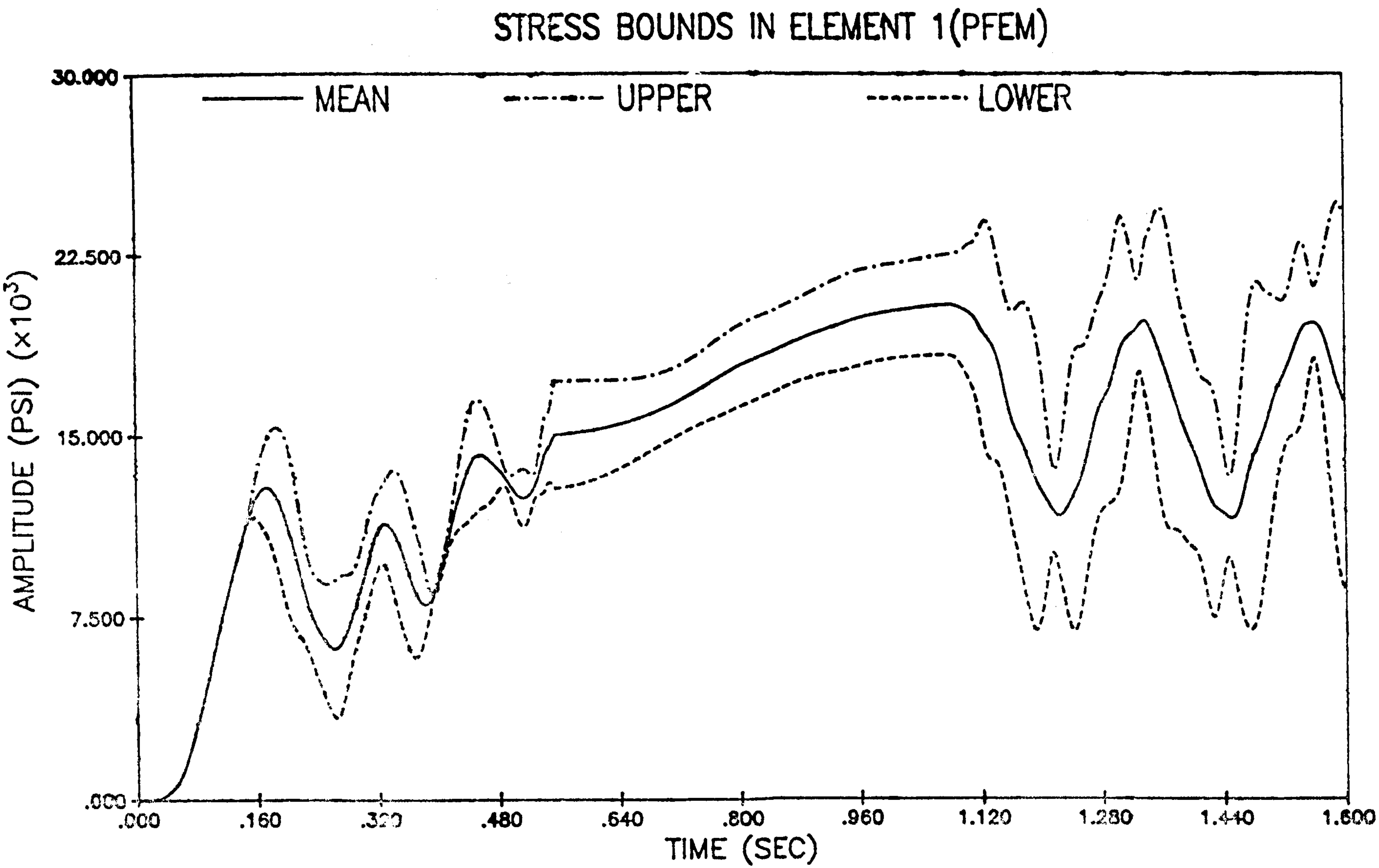
SLIDE 13

This shows the variance in the displacement at node 1 for this nonlinear example. In contrast to the linear problem, the agreement of the three methods is much better because the canonical terms have little effect in nonlinear response.


DISPLACEMENT BOUNDS AT NODE 1(PFEM)
MEAN
UPPER
LOWER
AMPLITUDE (IN)
60.001
45.000
30.000
15.000
.000
.000
.160
.320
.480
.640
.800
.960
1.120
1.280
1.440
1.600
TIME (SEC)

SLIDE 14

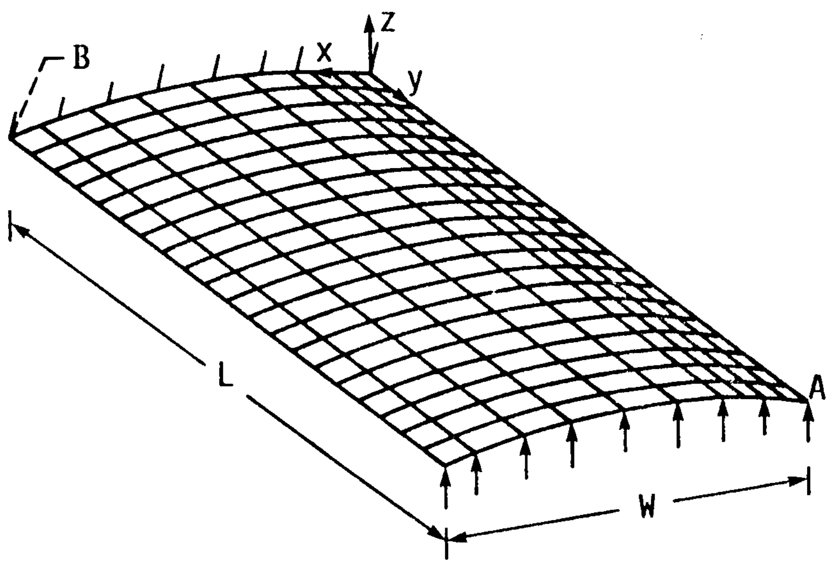
This shows the way displacement output would be presented to a typical user of a probabilistic finite element method computer program. The upper and lower bounds here are the displacements which are three standard deviations from the mean. As can be seen from these results, in nonlinear problems of this type, the effects of small variations in the yield stress on displacement results can be very severe.


STRESS BOUNDS IN ELEMENT 1(PFEM)
MEAN
UPPER
LOWER
30.000
22.500
15.000
7.500
.000
AMPLITUDE (PSI) (×10³)
.000
.160
.320
.480
.640
.800
.960
1.120
1.280
1.440
1.600
TIME (SEC)

SLIDE 15

This slide shows a similar representation for the stress in element no. 1. This can be seen from the time history of the response; until plasticity is initiated at about .15 seconds, all of the responses are the same, because the only randomness is in the yield stress. Subsequently, the method predicts a reasonably large range in the stresses.

CANTILEVERED SHELL: "TURBINE BLADE SIMULATION"



$E = 30.0 \times 10^6$
$E_t = 30.0 \times 10^4$
$\sigma_y = 25000.0$
(ISOTROPIC HARDENING)
$\nu = 0.3$ IN.
$L = 6.3$ IN.
$R = 2.29$ IN.

4 NODE SHELL ELEMENT
WITH SRI
462 NODES, 200 ELEMENTS
DISP. POINT A
STRESS POINT B
MAXIMUM THICKNESS
= 0.3 IN.

RANDOM LOAD CHARACTERISTICS

SIZE OF RANDOM LOAD VECTOR (q) = 10
COEFFICIENT OF VARIATION = 0.10
CORRELATION LENGTH (λ) = 4W
MAXIMUM MEAN LOAD = 13.25 LB

RANDOM MATERIAL CHARACTERISTICS

SIZE OF RANDOM MATERIAL VECTOR (q) = 20
COEFFICIENT OF VARIATION = 0.10
CORRELATION LENGTH (λ) = 4L
MEAN YIELD STRESS = 25000.0

SPATIAL CORRELATION OF RANDOM LOAD AND YIELD STRESS

$R(x_i, x_j) = \mathrm{EXP}(-\mathrm{ABS}(x_i - x_j)/\lambda)$

RANDOM LENGTH

1 VARIABLE, COV = 0.02

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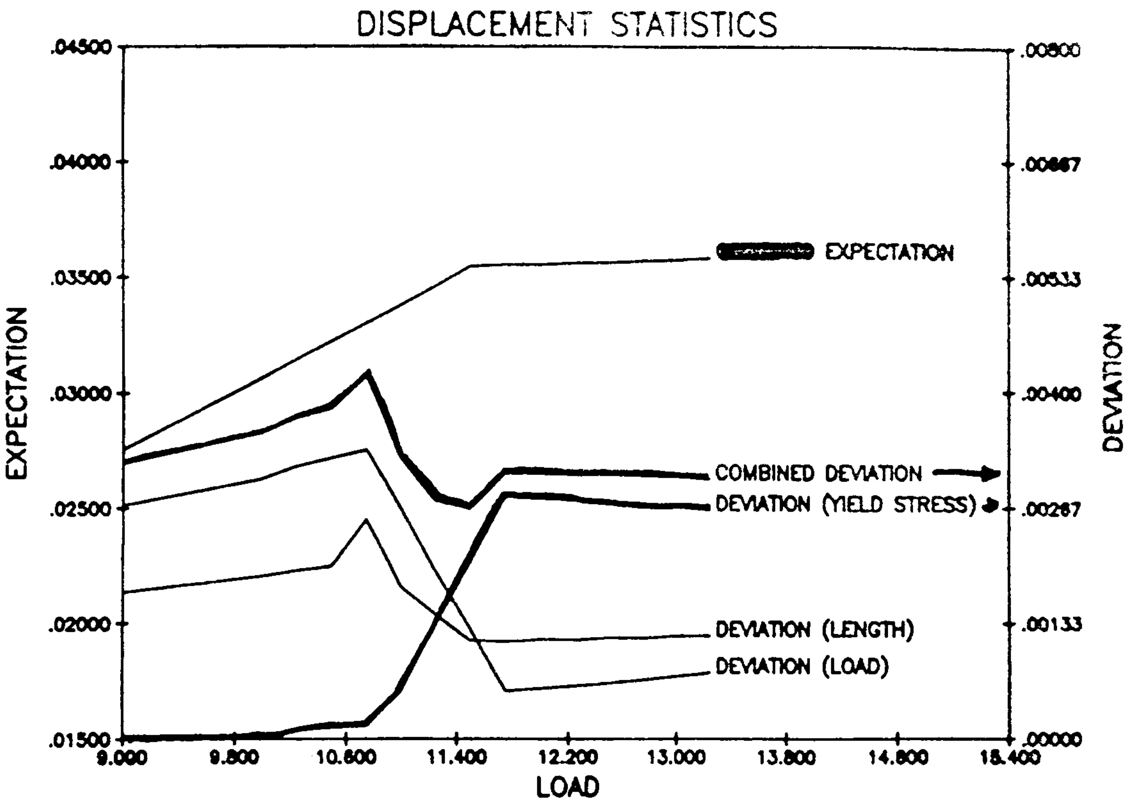
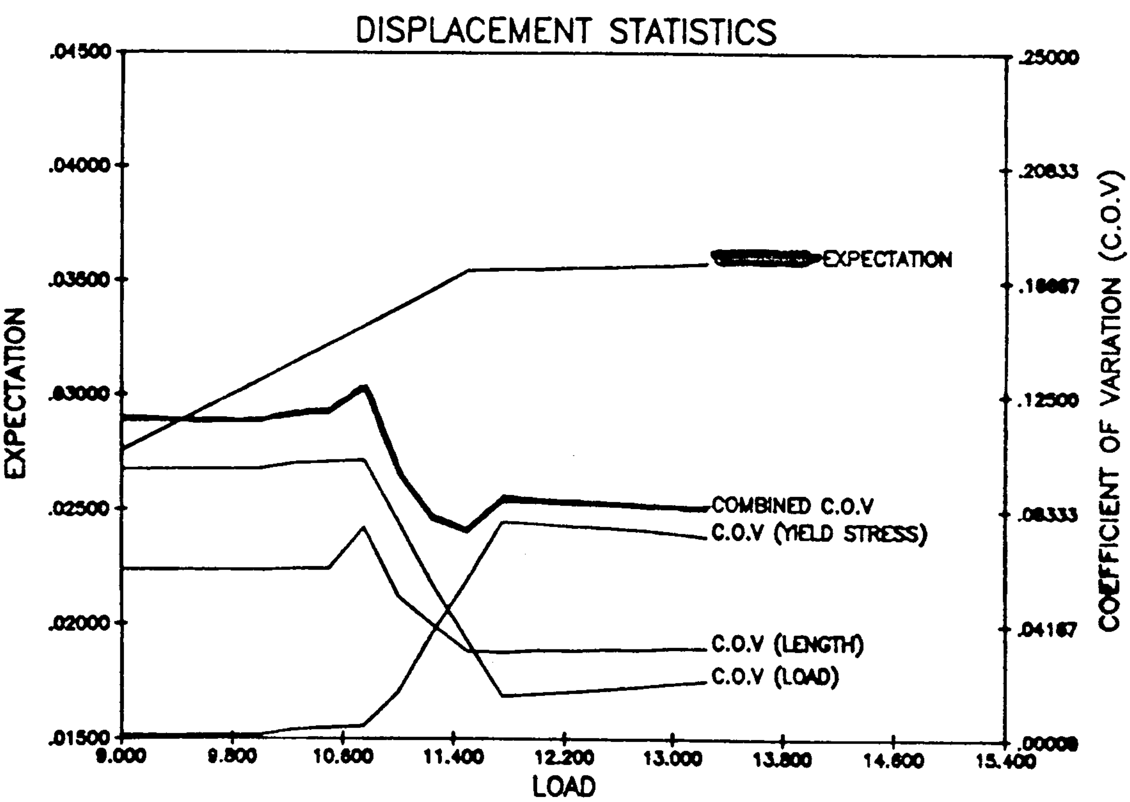
SLIDE 16

This figure shows a simplified model of a turbine blade which has randomness both in the geometry, as reflected in the length of the blade, and in the material properties, where the yield stress is random. In addition, the load, which is applied at the tip, is a random function.



DISPLACEMENT STATISTICS
EXPECTATION
DEVIATION
LOAD
COMBINED DEVIATION
DEVIATION (YIELD STRESS)
DEVIATION (LENGTH)
DEVIATION (LOAD)
.04500
.04000
.03500
.03000
.02500
.02000
.01500
.00533
.00400
.00267
.00133
.00000
9.000
9.800
10.600
11.400
12.200
13.000
13.800
14.800



DISPLACEMENT STATISTICS
EXPECTATION
COEFFICIENT OF VARIATION (C.O.V)
LOAD
COMBINED C.O.V
C.O.V (YIELD STRESS)
C.O.V (LENGTH)
C.O.V (LOAD)
.04500
.04000
.02500
.02000
.01500
.25000
.20833
.12500
.08333
.04167
9.000
9.800
10.600
11.400
12.200
13.000
13.800
14.600
15.400

SLIDE 17

SLIDE 17

This slide shows the expected value of the displacement at node a, labeled "expectation", and the variation of the displacement at that point. In addition to the variation which results from combined effects of randomness in load, length, and yield stress, the effects of the randomness in these three parameters are considered separately. It can be seen that randomness in the load has a major effect during the initial portion of the response, which is elastic, but its effect diminishes later. By contrast, the yield randomness in the yield stress becomes the dominant factor later in the response when the behavior of the turbine blade is elastic-plastic.

COMPUTATION TIME (Hu-Washizu V.P.)

- Plane Strain Beam Under Large Deflection
- 205 Nodes/160 4-Node Elements, 12 Load Steps
- Random Load, Material and Height

PHWVP Approach to PFEM

81 Corr. R.V.'s to 9 Uncorr. R.V.'s

1.5 cpu hrs. Total

Monte Carlo Simulation (MCS)

100 Samples: 60 cpu hrs.

400 Samples: 240 cpu hrs. (Projected)

Savings

PHWVP $\rightarrow$ MCS 100, 400: 97.5%, 99.4%

Note: Zeroth Order Eqn. is Nonlinear Whereas the First and Second Orders are Linear

SLIDE 18

To provide some guidelines as to why Monte Carlo simulation cannot be used for problems of this type, we have given the computation times for the PFEM method and Monte Carlo simulation for a 205-node, 160-element problem. The PFEM approach on a Harris H800 required about 1.5 cpu hours. By contrast, a 100-sample simulation requires 60 cpu hours. One hundred samples would probably not be sufficient to obtain reliable bounds on ±3 standard deviations, but increasing the number of samples to 400 would require 240 cpu hours.

PROPOSED WORK

- APPLY PFEM TO A STRAIGHT CRACK IN A "STRUCTURE" (BEAM, TURBINE BLADE) TO OBTAIN STATISTICS ON CRACK GROWTH
- ADD FIRST ORDER RELIABILITY ANALYSIS
- DEVELOP METHODOLOGIES FOR CRACKS THAT DO NOT GROW RECTILINEARLY AND FOR ELASTO-PLASTIC FRACTURE MECHANICS

SLIDE 19

This slide summarizes the work which is proposed in the continuation of this grant. The major objective of the work in the next year will be to incorporate a crack element into the probabilistic finite element program, so that the effects of randomness in the crack and load can be studied for a low-cycle fatigue-type problem. For this purpose, a straight crack will be incorporated in the program, and its growth will be predicted for a given random load.

A key feature of this development will be to incorporate the effect of the feedback between the actual structural configuration and the stress intensity at the crack into the growth model for the crack. Most current work for crack growth under random load or in random materials assumes the crack to be in an infinite medium, so that the effects of the structural configuration on the growth of the crack are not properly represented.

In these developments, in addition to the second-order moment methods which we have used for nonlinear structural dynamics, we will incorporate a first-order reliability analysis so that the probability of the crack's exceeding a certain threshold will be computable.

In subsequent years, we will extend these methods so that we can deal with cracks which do not grow in a straight line and to incorporate elasto-plastic fracture mechanics.